Chapter 8

DESIGN OF IIR FILTERS

8-1. INTRODUCTION

In this chapter we shall discuss the design of digital filters that have infinite-impulse-response durations. This type of filter is called an IIR filter. An IIR filter can be specified by an impulse response \( \{h(n), n = 0, 1, 2, \ldots\} \), a difference equation, or a transfer function. Among them, the transfer function is the easiest to use in the design. The most general form of the transfer function of a causal filter can be written as

\[
H(z) = \frac{b_0 z^N + b_1 z^{N-1} + b_2 z^{N-2} + \ldots + b_N}{z^N + a_1 z^{N-1} + a_2 z^{N-2} + \ldots + a_N}
\]

where not all \( a_i \) are equal to zero. We shall assume that there is no common factor between the denominator and numerator of \( H(z) \). The design problem is to find \( a_i \) and \( b_i \) so that \( H(z) \) satisfies design specifications.

As discussed in Chap. 6, the complete specification of a digital filter should consist of an amplitude characteristic, a phase characteristic, and a transient performance. In the actual design, however, because of the requirement of stability, causality, and simplicity, we often specify only an amplitude characteristic.
Hence, the design problem reduces to a search of $a$ and $b$ so that $|H(e^{j\omega T})|$ satisfies the amplitude characteristic. The amplitude of $H(e^{j\omega T})$ often consists of a square root of functions of $\omega$, and is not easy to handle. Thus, we prefer to compute $H(e^{j\omega T})H^*(e^{j\omega T}) = |H(e^{j\omega T})|^2$, and then check whether it meets the square of the amplitude characteristic. Clearly, the square of the amplitude characteristic can be easily obtained from the amplitude characteristic and will be called the amplitude-square characteristic. The design procedure then consists of two steps: search a function $B(\omega)$ that meets the amplitude-square characteristic, and then find a stable $H(z)$ so that

$$B(\omega) = H(z)H(z^{-1}) \bigg|_{z = e^{j\omega T}} = |H(e^{j\omega T})|^2 \quad (8-1)$$

Of course, not every $B(\omega)$ can be decomposed as in Eq. (8-1). We call $B(\omega)$ decomposable if it can be so decomposed. Therefore, the design problem is to find a decomposable $B(\omega)$ to meet the amplitude-square characteristic. In the digital case, it turns out that the conditions for $B(\omega)$ to be decomposable must be stated in terms of trigonometric functions, and is rather complicated; therefore, it is not very easy to design a digital filter by using the above procedure.

In this chapter, IIR digital filters will be designed from analog filters by using various transformations. There are three reasons for taking this approach. First, the conditions of decomposability in the analog case are very simple; therefore, analog filters can be easily designed. Second, the design of analog filters is complete and well documented. Finally, the effort to design a digital filter from an analog one may be less than the one from direct design. In addition, we also learn the design of analog filters.

8-2. DESIGN OF ANALOG FILTERS

In this section we shall study the design of analog filters. Similar to the digital case, we study only linear, time-invariant, and causal filters. This type of filter can be described by the continuous
convolution [see Eq. (2-61)], the transfer function, and the differential equations. In the design we use only the transfer function. Thus, we discuss only this description.

Consider a function of time \( f(t) \) defined for \( 0 \leq t < \infty \). The Laplace transform of \( f(t) \), denoted by \( \mathcal{L}[f(t)] \), is defined as

\[
F(s) \triangleq \mathcal{L}[f(t)] = \int_0^\infty f(t)e^{-st} \, dt
\]  

(8-2)

where \( s \) is a complex variable. The role of \( s \) in the Laplace transform is similar to the role of \( z \) in the z-transform. As an example, the Laplace transform of \( f(t) = e^{\lambda t} \), for \( t \geq 0 \) and \( \lambda \) is a real or complex number, is

\[
F(s) = \int_0^\infty e^{\lambda t}e^{-st} \, dt = \frac{1}{-(s - \lambda)} e^{-(s-\lambda)t} \bigg|_{t=0}^{t=\infty}
\]  

(8-3)

If \( \text{Re } s > \text{Re } \lambda \), where \( \text{Re } \) stands for the real part, then \( e^{-(s-\lambda)t} = 0 \) at \( t = \infty \). Hence, Eq. (8-3) becomes

\[
F(s) = \frac{1}{-(s - \lambda)} (0 - 1) = \frac{1}{s - \lambda}
\]

and \( \text{Re } s > \text{Re } \lambda \) constitutes the region of convergence as shown in Fig. 8-1. If we write \( s = \sigma + j\omega \) and if the region of convergence

![Diagram](image)

FIG. 8-1. The region of convergence of the Laplace transform of \( f(t) = e^{\lambda t}, \, t \geq 0 \).
includes the $j\omega$-axis, then the Laplace transform with the replacement of $s = j\omega$ is identical to the Fourier transform. Thus, what is discussed for the Fourier transform is directly applicable to the Laplace transform.

In terms of the Laplace transform, the input $u(t)$ and the output $y(t)$ of any linear, time-invariant, causal analog filter that is relaxed at $t = 0$ can be related by, as in (5-6) for the digital case,

$$Y(s) = H(s)U(s)$$  \hspace{1cm} (8-4)

where $Y(s)$ and $U(s)$ are, respectively, the Laplace transforms of $y(t)$ and $u(t)$ [see Eq. (2-63)]. The function $H(s)$ is called the transfer function of the filter. We see that this equation is identical to the transfer-function description of a digital filter if $s$ is replaced by $z$. Similar to the digital case, we may define an analog filter to be stable if every bounded input excites a bounded output. The condition for an analog filter to be stable is that all the poles of $H(s)$ have negative real parts, or equivalently, all the poles of $H(s)$ lie inside the left-half $s$-plane. Recall that the stability condition for discrete-time systems is that all poles be inside the unit circle. Although the stability is defined as bounded-input bounded-output for both cases, the stability conditions are different. Actually this is the only major difference between analog and digital systems. Similar to the stability test in Sec. 5-6, there is a stability test, called the Routh-Hurwitz criterion, available to check the stability of analog filters without computing the roots. Since this test is not used in this text, its discussion is omitted. The interested reader is referred to Chen (1970, 1976).

If we apply a sinusoidal input \( u(t) = \sin \omega_0 t, \ t \geq 0 \) to a stable analog filter, then the steady-state output of the filter is given by, as in (5-28),

$$y_{ss}(t) = |H(j\omega_0)| \sin (\omega_0 t + \angle H(j\omega_0))$$  \hspace{1cm} (8-5)

Hence, as in the digital case, by specifying the amplitude characteristic \( |H(j\omega)| \) and the phase characteristic \( \angle H(j\omega) \), we can obtain various types of analog filters. For example, if an analog filter
FIG. 8-2. The amplitude and phase characteristics of an ideal low-pass analog filter.

has the characteristics shown in Fig. 8-2, then any analog signal with frequency spectrum lying inside \([0, \omega_p]\) can pass through the filter without any attenuation. On the other hand, any analog signal with frequency spectrum lying outside \([0, \omega_p]\) will be blocked completely by the filter. This type of filter is called an ideal low-pass filter. Unfortunately, this type of filter is not physically realizable or, equivalently, not causal. Even if we can design a causal filter to approximate the ideal low-pass filter, the transfer function of the filter will be an irrational function or a rational function of a very high degree. In order to have a simple transfer function, we generally specify the amplitude characteristic as shown in Fig. 8-3. Therefore, the design problem reduces to the

FIG. 8-3. The specification of a low-pass analog filter.
the search of a stable, proper transfer function $H(s)$ of a relatively small degree to meet the specification shown. Once such a transfer function is found, we shall check its phase characteristic and transient performance such as rise time, settling time, and overshoot that can be defined similarly as in the digital case. The situation here is identical to the design of digital filters.

We discuss first the design of low-pass analog filters. The design procedure consists of two parts: first, find a $B(\omega)$ to meet the amplitude-square characteristic, and then find a stable $H(s)$ such that

$$B(\omega) = H(s)H(-s) \bigg|_{s=j\omega} = |H(j\omega)|^2$$

(8-6)

It is clear that not every $B(\omega)$ can be decomposed as in (8-6). Fortunately, the condition for $B(\omega)$ to be decomposable is very simple: $B(\omega)$ must be an even function of $\omega$, a proper rational function of $\omega$, and positive for all $\omega$.

We discuss in this section two types of low-pass analog filters, namely, Butterworth and Chebyshev filters. Though elliptic filters are also widely used in practice, the discussion requires an introduction of elliptic functions and their tables. Thus, discussion of them is omitted.

Butterworth Filter

The amplitude characteristic of a Butterworth filter can be written as

$$B(\omega) = |H(j\omega)|^2 = \frac{1}{1 + (\omega/\omega_p)^{2n}}$$

(8-7)

where $n$ is a positive integer, and $\omega_p$ is the band-pass cutoff.

+A rational function is called proper if the degree of its denominator is equal to or larger than that of its numerator (see Definition 5-2). It can be established that every proper rational function $H(s)$ can be realized as a causal analog filter. [See Papoulis (1962).]
frequency and is defined as the frequency at which $|H(j\omega_p)| = 0.707$ or, equivalently, $20 \log |H(j\omega_p)| = -3$ dB. The characteristic of Eq. (8-7) is plotted in Fig. 8-4 for various $n$. We see that the amplitude is monotonically decreasing in both the pass band and stop band. In fact, this type of filter is flattest at $\omega = 0$ in the sense that $\frac{d^iH(j\omega)}{d\omega^i} = 0$ for $i = 1, 2, \ldots, m$, with the
largest \( m \) among all transfer functions with constant numerators. Hence, a Butterworth filter is also called a \textit{maximally-flat} filter. Clearly, \( B(\omega) \) is a positive, proper, and even function of \( \omega \); hence, it is decomposable.

The integer \( n \) in (8-7) is determined by the specification of the stop band. Suppose it is required that for \( \omega \geq \omega_q \), the amplitude be smaller than \( a \), or equivalently, the attenuation be larger than \(-20 \log a \) dB, then \( n \) must be chosen to be the smallest integer such that

\[
\frac{1}{1 + (\omega_q/\omega_p)^{2n}} \leq a^2 \tag{8-8}
\]

The integer \( n \) can also be read from the plot in Fig. 8-5 in which the frequency is in the logarithmic scale. For example, if the attenuation is required to be larger than 40 dB for \( \omega \geq 2 \omega_p \), then \( n \) should be larger than 7. Hence, from the specification of the cut-off frequencies \( \omega_q \) and \( \omega_p \) and the tolerance in the stop band, we can specify completely \( |H(j\omega)|^2 \).

The next step in the design is to find a stable transfer function \( H(s) \) that satisfies (8-6). The substitution of \( \omega = s/j \) into (8-7) yields

\[
|H(s)|^2 = \frac{1}{1 + (-1)^n(s/\omega_p)^{2n}} \tag{8-9}
\]

There are \( 2n \) roots in \( 1 + (-1)^n(s/\omega_p)^{2n} = 0 \). They are given by

\[
\left(\frac{s}{\omega_p}\right)^{2n} = (-1)^{1-n} e^{j[\pi(1-n)+2\pi k]} \quad k = 0, 1, 2, \ldots, 2n - 1
\]

or

\[
s = \omega_p e^{j\pi(1-n+2k)/2n} \quad k = 0, 1, 2, \ldots, 2n - 1
\]

The \( 2n \) roots are equally distributed on the circle with radius \( \omega_p \) and apart by \( 2\pi/2n \) rad. If \( n \) is an odd integer, the pole starts from 0 rad; if \( n \) is an even integer, then the pole starts from \( \pi/2n \) rad, as shown in Fig. 8-6. We see that the distribution of the
poles is symmetric with respect to the jω-axis. Hence, if \( H(s) \) is chosen to consist of all the poles inside the left-half s-plane, then \( H(-s) \) will consist of the poles in the right-half s-plane. Furthermore, the \( H(s) \) so chosen is stable. The poles inside the left-half s-plane are

\[
s_i = \omega_p \left[ -\sin \left( \frac{(2i + 1)\pi}{2n} \right) + j \cos \left( \frac{(2i + 1)\pi}{2n} \right) \right]
\]

\[i = 0, 1, \ldots, n - 1\] (8-10)

They can be readily derived from Fig. 8-6. Hence, the transfer function of a Butterworth filter is

\[
H(s) = \frac{1}{(-1)^n \prod_{i=0}^{n-1} (s/s_i - 1)}
\]

where \( s_i \) are given in (8-10). The factor \((-1)^n\) is to insure \( H(0) = 1 \). For convenience of use, we list in Table 8-1

\[(-1)^n \prod_{i=0}^{n-1} (s/s_i - 1)\] with \( \omega_p = 1 \) and \( n = 1, 2, \ldots, 8 \). Using this table, the transfer function of a Butterworth filter can be readily obtained. Note that these formulas are listed for \( \omega_p = 1 \). If \( \omega_p \neq 1 \), then \( s \) must be replaced by \( s/\omega_p \). For example, the Butterworth filter of degree 3 with bandwidth 2 rad/sec is

\[
\frac{1}{(s/2 + 1)[(s/2)^2 + s/2 + 1]} = \frac{8}{(s + 2)(s^2 + 2s + 4)}
\]
TABLE 8-1. The Denominator of Butterworth Filters with \( \omega_p = 1 \) rad/sec.

<table>
<thead>
<tr>
<th>( n )</th>
<th>(-1)^n ( \prod_{i=0}^{n-1} \left( \frac{s - s_i}{s_i} \right) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>( s + 1 )</td>
</tr>
<tr>
<td>2</td>
<td>( s^2 + 1.414s + 1 )</td>
</tr>
<tr>
<td>3</td>
<td>((s + 1)(s^2 + s + 1))</td>
</tr>
<tr>
<td>4</td>
<td>((s^2 + 0.7653s + 1)(s^2 + 1.8477s + 1))</td>
</tr>
<tr>
<td>5</td>
<td>((s + 1)(s^2 + 0.6180s + 1)(s^2 + 1.6180s + 1))</td>
</tr>
<tr>
<td>6</td>
<td>((s^2 + 0.5176s + 1)(s^2 + 1.414s + 1)(s^2 + 1.9318s + 1))</td>
</tr>
<tr>
<td>7</td>
<td>((s + 1)(s^2 + 0.4449s + 1)(s^2 + 1.2465s + 1)) (s^2 + 1.8022s + 1)</td>
</tr>
<tr>
<td>8</td>
<td>((s^2 + 0.3896s + 1)(s^2 + 1.1110s + 1)(s^2 + 1.6630s + 1)) (s^2 + 1.9622s + 1)</td>
</tr>
</tbody>
</table>

Example 1 Design an analog filter that is monotonic in the pass band and stop band. The bandwidth, defined for 3-dB attenuation, of the filter is required to be 5 rad/sec. The attenuation should be at least 30 dB for frequencies larger than 15 rad/sec.

We use a Butterworth filter. From Fig. 8-5, we see that at 15/5 the attenuation is about 29 dB for \( n = 3 \) and 38 dB for \( n = 4 \). Hence, we choose \( n = 4 \), and the transfer function of the required filter is given by

\[
H(s) = \frac{1}{(s^2 + 0.7653s + 1)(s^2 + 1.8477s + 1)}
\]

\[
\frac{625}{(s^2 + 3.8265s + 25)(s^2 + 9.2385s + 25)}
\]

\[
\frac{625}{s^4 + 13.066s^3 + 85.355s^2 + 326.638s + 625}
\]  

(8-12)

Though the Butterworth filter is very easy to design, the rate at which its amplitude decreases in the frequency range \( \omega \geq \omega_p \) is rather slow for small \( n \). Consequently, for a given transition band, the degree of the Butterworth filter required is often larger than that of other types of filters. Furthermore, for large \( n \), the
overshoot of the step response of a Butterworth filter is rather large (over 12 percent). Thus, in some applications, we wish to use other types of filters.

Chebyshev Filters

Let \( V_n(x) \) be the Chebyshev polynomial of order \( n \) defined as

\[
V_{n+1}(x) = 2xV_n(x) + V_{n-1}(x) = 0 \tag{8-13}
\]

with

\[
V_1(x) = x \tag{8-14}
\]

and

\[
V_2(x) = 2x^2 - 1 \tag{8-15}
\]

Using this recursive equation, we can compute

\[
\begin{align*}
V_3(x) &= 4x^3 - 3x \\
V_4(x) &= 8x^4 - 8x^2 + 1 \\
V_5(x) &= 16x^5 - 20x^3 + 5x \\
V_6(x) &= 32x^6 - 48x^4 + 18x^2 - 1 \\
V_7(x) &= 64x^7 - 112x^5 + 56x^3 - 7x \\
V_8(x) &= 128x^8 - 256x^6 + 160x^4 - 32x^2 + 1
\end{align*}
\tag{8-16}
\]

and so forth. We note that \( V_n(1) = 1 \) for all \( n \). In terms of \( V_n(x) \), we define

\[
B(\omega) = \left| H(j\omega) \right|^2 = \frac{1}{1 + \epsilon^2 \frac{V^2_n(\omega/\omega_p)}{2}} \tag{8-17}
\]

where \( \epsilon \) is a positive number. The characteristic of Eq. (8-17) is plotted in Fig. 8-7. We see that it is equiripple in the range \([0,1]\), and monotonically decreasing for \( \omega \geq \omega_p \). A filter with this type of characteristics is called a Chebyshev filter. It can be shown that this type of filter has the steepest possible cutoff rate in the transition band among all transfer functions with constant numerators.
Fig. 8-7. The characteristics of Chebyshev filters.

We discuss more the plots of $|H(j\omega)|^2$ in (8-17). Since $V_n(\omega/\omega_p) = 1$ at $\omega = \omega_p$ for all $n$, the magnitude of $H(j\omega)$ at $\omega = \omega_p$ is equal to $(1 + \varepsilon^2)^{-0.5}$. The magnitude of $H(j\omega)$ at $\omega = 0$ is equal to 1 if $n$ is odd and equal to $(1 + \varepsilon^2)^{-0.5}$ if $n$ is even. Unlike the Butterworth filter in which the pass-band cutoff frequency is defined for $|H(j\omega_p)| = 0.707$, the pass-band cutoff frequency of a Chebyshev filter is defined for the largest $\omega_p$ such that $|H(j\omega_p)| = (1 + \varepsilon^2)^{-0.5}$. The magnitude of the ripple is clearly equal to

$$\delta = 1 - (1 + \varepsilon^2)^{-0.5} \quad (8-18)$$

Hence, from the specification of the ripple, the constant $\varepsilon$ in Eq. (8-18) can be computed.

The integer $n$ in (8-17) is determined, as in the Butterworth case, by the specification of the stop band. For example, if it is
required that for \( \omega \geq \omega_q \), the magnitude be smaller than \( a \), then \( n \) should be chosen to be the smallest integer such that

\[
\frac{1}{1 + \varepsilon^2 V^2_n(\omega/\omega_p)} \leq a^2
\]  

(8-19)

Solving of this inequality, though tedious, can be carried out in a straightforward manner. We plot in Fig. 8-8 the attenuation of Chebyshev filters for \( \omega \geq \omega_p \) and \( \varepsilon^2 = 0.023 \) or \( 10 \log \left[ 1/(1 + \varepsilon^2) \right] = -0.1 \text{ dB} \). If the ripple magnitude happens to be 0.1 dB, then the required \( n \) can be read directly from Fig. 8-8. Otherwise, we have to solve (8-19) for \( n \).

Once \( \varepsilon \) and \( n \) are determined, the next step is to find \( H(s) \) to satisfy \( H(s)H(-s) \big|_{s=j\omega} = B(\omega) \). If \( \omega \) in (8-17) is replaced by \( s/j \), then it can be shown that the \( 2n \) poles of

\[
|H(s)|^2 = \frac{1}{1 + \varepsilon^2 V^2_n(s/j\omega_p)}
\]

are given by

\( \omega_p \)

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig8-8}
\caption{Stop-band attenuation of Chebyshev filters with 0.1-dB pass-band ripples.}
\end{figure}


\[ s_i = \omega_p \left[ -b \sin \left( \frac{(2i + 1)\pi}{2n} \right) + jc \cos \left( \frac{(2i + 1)\pi}{2n} \right) \right] \]

\[ i = 0, 1, 2, \ldots, 2n - 1 \]  

(8-20)

where

\[ m = \left( \sqrt{1 + \varepsilon^2} + \varepsilon^{-1} \right)^{1/n} \]  

(8-21)

and

\[ b = \frac{m - m^{-1}}{2} \quad c = \frac{m + m^{-1}}{2} \]  

(8-22)

If a table of hyperbolic functions is available, \( b \) and \( c \) can also be obtained as

\[ b = \sinh \phi \]  

(8-23)

\[ c = \cosh \phi \]  

(8-24)

where

\[ \phi = \frac{1}{n} \sinh^{-1} \frac{1}{\varepsilon} \]  

(8-25)

Note that \( s_i, i = 0, 1, 2, \ldots, n - 1 \) are located in the left-hand \( s \)-plane, and the rest in the right-half \( s \)-plane. By comparing (8-10) and (8-20), we see that the poles of the Chebyshev filter is related to those of the Butterworth filter. In fact, the former can be obtained from the latter as shown in Fig. 8-9. We draw two circles with radii \( bw \) and \( cw \). We then plot the poles of the Butterworth filter on both circles. The intersection of the horizontal line drawn from the outer pole with the vertical line drawn from the corresponding inner pole gives a pole of the Chebyshev filter. We see that the poles of (8-17) are symmetric with respect to the imaginary axis of the \( s \)-plane; hence, the transfer function of the Chebyshev filter is

\[
H(s) = \begin{cases} 
\prod_{i=0}^{n-1} \left( \frac{s}{s_i} - 1 \right) & \text{n:odd} \\
\sqrt{1 + \varepsilon^2} \prod_{i=0}^{n-1} \left( \frac{s}{s_i} - 1 \right) & \text{n:even}
\end{cases}
\]
where the $s_i$ are given in (8-20). It consists of all the left-half $s$-plane poles of $B(s/j)$.

**Example 2** Design an analog filter that is equiripple in the pass band and monotonic in the stop band. The bandwidth defined for 3-dB attenuation is required to be 5 rad/sec. The attenuation should be at least 30 dB for $\omega \geq 15$ rad/sec.

We use the Chebyshev filter defined by (8-17). The $\varepsilon^2$ in (8-17) should be chosen so that

$$10 \log \frac{1}{1 + \varepsilon^2 V_n^2(1)} = -3$$

which implies $\varepsilon^2 = 1$. The integer $n$ must satisfy

$$10 \log \frac{1}{1 + V_n^2(15/5)} \leq -30 \quad \text{or} \quad 1 + V_n^2(3) \geq 10^3$$

We compute

$$V_n^2(3) = 9$$
$$V_n^2(3) = 17^2 = 289$$
$$V_n^2(3) = (4 \times 27 - 9)^2 = 99^2 = 9801$$
Thus, we choose \( n = 3 \). We now compute

\[
\begin{align*}
b &= \frac{2.414^{1/3} - 2.414^{-1/3}}{2} = \frac{1.3415 - 0.7454}{2} = 0.298 \\
c &= \frac{2.414^{1/3} + 2.414^{-1/3}}{2} = 1.043
\end{align*}
\]

Hence, the poles of the required Chebyshev filter are

\[
\begin{align*}
s_0 &= 5(-0.298 \sin 30° + j1.043 \cos 30°) = -0.745 + j4.516 \\
s_1 &= 5(-0.298 \sin 90° + j1.043 \cos 90°) = -1.49 \\
s_2 &= 5(-0.298 \sin 150° + j1.043 \cos 150°) = -0.745 - j4.516
\end{align*}
\]

and the transfer function of the filter is

\[
H(s) = \frac{(-1)^3}{(s/s_0 - 1)(s/s_1 - 1)(s/s_2 - 1)}
\]

\[
= \frac{31.214}{(s + 1.49)(s + 0.745 + j4.516)(s + 0.745 - j4.516)}
\]

\[
= \frac{31.214}{s^3 + 2.98s^2 + 23.169s + 31.214}
\] (8-26)

The Chebyshev filter introduced is one of two possible forms. The other form has ripple in the stop band and is monotonic in the pass band. This form is not as useful as the one introduced in this section for the design of digital filters, and its discussion is omitted.

8-3. ANALOG FREQUENCY TRANSFORMATIONS

We discussed in the previous section the design of analog low-pass filters. Although the same procedure can be applied to design band-pass, high-pass, or band-stop filters, it is much easier to obtain these filters from a low-pass filter by using frequency transformations. In this section, these transformations will be introduced.