Circuit Theorems: Thevenin and Norton Equivalents, Maximum Power Transfer

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Thevenin’s Theorem

- Any circuit with sources (dependent and/or independent) and resistors can be replaced by an equivalent circuit containing a single voltage source and a single resistor.
- Thevenin’s theorem implies that we can replace arbitrarily complicated networks with simple networks for purposes of analysis.
Independent Sources (Thevenin)

Circuit with independent sources

Thevenin equivalent circuit

$V_{Th}$

$R_{Th}$
No Independent Sources

Circuit without independent sources

$R_{Th}$

Thevenin equivalent circuit
Thevenin Equivalent Circuit

- Basic steps to determining Thevenin equivalent are
  - Find $v_{Th}$

$V_{oc} = V_{Th}$
- Compute the Thevenin equivalent resistance, $R_{Th}$

  (a) If there are only independent sources, then short circuit all the voltage sources and open circuit the current sources (just like superposition).

\[
R_{in} = R_{Th}
\]
(b) If there are only dependent sources, then must use a test voltage or current source in order to calculate

\[ R_{Th} = \frac{V_{test}}{I_{test}} \]
Thevenin Equivalent Circuit

(c) If there are both independent and dependent sources, then compute

(i) \( R_{Th} = \frac{V_{Test}}{I_{test}} \) (all independent sources set equal to zero)
(ii) compute \( R_{Th} \) from \( V_{OC}/I_{SC} \).

Linear Two-terminal circuit

\[ R_{Th} \]

\[ i = i_{sc} \]

\[ V_{Th} \]

\[ i = i_{sc} \]

\[ i_{sc} = \frac{V_{Th}}{R_{Th}} \]
Example

Find I using Thevenin’s Theorem

[Diagram of an electrical circuit with nodes labeled 2Ω, 2Ω, 5 V, 3 A, and 4Ω, and an arrow indicating the direction of current I.]
Step 1: Get the Thevenin Equiv. of the circuit to the left of terminals a-b
Step 1a: Open circuit voltage calculation

KCL at $V_{oc}$: \[
\frac{V_{oc}}{2} + \frac{V_{oc} - 5}{2} = 3 \quad \Rightarrow \quad V_{oc} = 5.5V
\]
Example cont.

Step 1b: Determination of $R_{Th}$

$R_{Th} = 2\Omega \parallel 2\Omega = 1\Omega$

0 A.
Example cont.

$I = \frac{5.5 \text{ V.}}{(1 + 4) \Omega} = 1.1 \text{ A.}
(\text{Ohm’s Law})$

$R_{Th} = 1 \Omega$

$V_{Th} = 5.5 \text{ V.}$
Problem: for the following circuit, determine the Thevenin equivalent circuit.
Solution:

Step 1: In this circuit, we have a dependent source. Hence, we start by finding the open circuit voltage $V_{oc} = V_{ab}$.

- KCL at node C
- $(5 - V_{oc})/2 + V_{oc}/4 = 0$
- $V_{oc} = 10$ V
Step 2: We obtain the short circuit current \( I_{sc} \) by shorting nodes a-b and finding the current through it.

\[ 5 = 2 \cdot I_{sc} + 3 \cdot I_{sc} \quad \Rightarrow \quad I_{sc} = \frac{5}{5} \]

\( I_{sc} = 1 \text{ A} \)
Step 3: Find the equivalent Thevenin Voltage and Resistance

- $V_{th} = V_{oc} = V_{ab} = 10V$
- $R_{th} = \frac{V_{oc}}{I_{sc}} \Rightarrow R_{th} = \frac{10}{1} \, \Omega$
- $V_{th} = 10V$
- $R_{th} = 10 \, \Omega$
Norton Equivalent Circuit

- Any Thevenin equivalent circuit is in turn equivalent to a current source in parallel with a resistor [source transformation].
- A current source in parallel with a resistor is called a Norton equivalent circuit.
Finding a Norton equivalent circuit requires essentially the same process as finding a Thevenin equivalent circuit.

\[ V_{Th} = R_N I_N \]
\[ I_N = \frac{V_{Th}}{R_{Th}} \]
\[ R_{Th} = R_N \]

- Finding a Norton equivalent circuit requires essentially the same process as finding a Thevenin equivalent circuit.
Thevenin/Norton Analysis

1. Pick a good breaking point in the circuit (cannot split a dependent source and its control variable).

2. Thevenin: Compute the open circuit voltage, $V_{OC}$.
   Norton: Compute the short circuit current, $I_{SC}$.

   If there is not any independent source then both $V_{OC}=0$ and $I_{SC}=0$ [so skip step 2]
Thevenin/Norton Analysis

3. Calculate $R_{Th}(R_N) = \frac{V_{oc}}{I_{sc}}$

4. **Thevenin**: Replace circuit with $V_{OC}$ in series with $R_{Th}$
   
   **Norton**: Replace circuit with $I_{SC}$ in parallel with $R_{Th}$

Note: for circuits containing no independent sources the equivalent network is merely $R_{Th}$, that is, no voltage (or current) source.

**Only steps 2 & 4 differ from Thevenin & Norton!**
Maximum Power Transfer
Maximum Power Transfer

Power delivered to the load as a function of $R_L$.
Maximum Power Transfer

\[
I = \frac{V_{Th}}{R_{Th} + R_L}
\]

\[
P_{R_L} = I^2 R_L = \frac{V_{Th}^2}{(R_{Th} + R_L)^2} \cdot R_L
\]

To find the maxima

\[
\frac{d P_{R_L}}{d R_L} = 0
\]

\[
\frac{d}{d R_L} \left[ \frac{V_{Th}^2 R_L}{(R_{Th} + R_L)^2} \right] = 0
\]

Note: \( d \left( \frac{u'}{v} \right) = \frac{u' v - u v'}{v^2} \)
Maximum Power Transfer

\[ V^2_{Th} \left( \frac{(R_{Th} + R_L)^2 \cdot 1 - R_L \cdot (2(R_{Th} + R_L))}{(R_{Th} + R_L)^2} \right) = 0 \]

\[ R^2_{Th} + R^2_L + 2R_{Th} \cdot R_L - 2R_{Th} \cdot R_L - 2R^2_L = 0 \]

\[ R^2_{Th} - R^2_L = 0 \]

\[ R_{Th} = R_L \leftarrow \text{Maximum Power Transfer} \]

\[ P_{\text{max}} = \left( \frac{V^2_{Th}}{(2R_L)^2} \right) \cdot R_L = \left( \frac{V^2_{Th}}{4R_L} \right) \text{ watts} \]
Example

What’s the maximum power that can be extracted from terminals a-b?
Example cont.

The circuit’s Thevenin equivalent loaded with $R_{Th}$ at terminals a-b yields:

$I = \frac{5.5 \text{ V.}}{(1 + 1) \Omega} = 2.75 \text{ A.}$ so the (maximum) load power is: $P_{\text{max.}} = I^2 R = (2.75 \text{ A.})^2 \times 1 \Omega = 7.5625 \text{ W.}$
Example

Determine the value of R in the circuit which will draw maximum power and calculate the corresponding maximum power.
First find $V_{th} =$ open-circuit voltage across terminals 1 & 3

KCL at the supernode:

\[
\frac{V_{th} + 17}{3} + \frac{V_{th} + 6}{2} = -2 \Rightarrow \frac{5}{6} V_{th} = -2 - \frac{17}{3} - 3
\]

\[
V_{th} = \frac{6}{5} \left( -\frac{32}{3} \right) = -\frac{64}{5} = -12.8V
\]
$R_{Th} = \text{Resistance across (open-circuited) terminals 1 \& 3 with the independent sources deactivated}$

The parallel combination of the 0 Ω SC and the 3 Ω resistor is 0 Ω (another SC) so the circuit becomes (next slide) ...
$R_{Th} = \text{Resistance across (open-circuited) terminals 1 \& 3 with the independent sources deactivated}$

$R_{Th} = 3 \, \Omega \ || 2 \, \Omega = 1.2 \, \Omega$
The circuit’s Thevenin equivalent loaded with $R = R_{Th}$ draws a current of:

$$I = \frac{V_{Th}}{R_{Th} + R} = \left( -12.8 \text{ V.} \right) / \left( 1.2 \ \Omega + 1.2 \ \Omega \right) = -5\frac{1}{3} \text{A.}$$

and the corresponding maximum power is

$$P_{max} = I^2R = \left( -5\frac{1}{3} \text{A.} \right)^2 \times 1.2 \ \Omega = 34\frac{2}{15} \text{ W.} \approx 34.13 \text{ W.}$$