Fourier Series Representation Of Periodic Signals &
The Continuous Time Fourier Transform

I- Computing the Discrete-Time Fourier series with \textit{fft}

The discrete-time Fourier series (DTFS) is a frequency-domain representation for periodic discrete-time sequences. For a signal \( x[n] \) with fundamental period \( N \), the DTFS synthesis and analysis equations are given by:

\[
x[n] = \sum_{k=0}^{N-1} X[k] e^{jk(2\pi/N)n} \quad (1.1)
\]

\[
X[k] = \frac{1}{N} \sum_{n=0}^{N-1} x[n] e^{-jk(2\pi/N)n} \quad (1.2)
\]

To compute Equations (1.1) and (1.2) if \( x \) is an \( N \)-point vector containing for the period \( 0 \leq n \leq N-1 \), then the DTFS of \( x[n] \) can be computed by \( X = (1/N) * \text{fft}(x) \). Where the \( N \)-point vector \( X \) contains \( X[k] \) for \( 0 \leq k \leq N-1 \). The function \textit{fft} is simply an efficient implementation of Eqn. (1.2) scaled by \( N \).

\textit{Example:}

\[
X[n]=\begin{cases} n & -2 \leq n \leq 2 \\ 0 & \text{otherwise} \end{cases}
\]

Assume \( x[n] \) is the signal with Fundamental period \( N=8 \). Define \( x \). Using MATLAB, plot the signal on the interval \(-2 \leq n \leq (2N-1)-2 \). The DTFS can be computed by typing \( X=(1/N)*\text{fft}(x) \). Plot the real and Imaginary parts of Fourier series coefficients. You can verify analytically that these are the correct values for \( x[k] \) \(-2 \leq k \leq (2N-1)-2 \); the function \textit{ifft} can be used to construct a vector \( x \) containing \( x[n] \) for \(-N \leq n \leq (2N-1)-2 \).

\textit{Answer:}

\begin{verbatim}
N=8;
n=-2:(2*N-1)-2;
k=[-2 -1 0 1 2];
x=[k zeros(1,3) k zeros(1,3)];
X=(1/N)*fft(x);
x1=N*ifft(X);
figure(1)
stem(n,x,'filled')
xlabel('time')
ylabel('Amplitude')
title('x(n)')
figure(2)
subplot(2,1,1),stem(n,real(X),'filled')
xlabel('k')
end
\end{verbatim}
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del('FS Real Part')
title('real part of X[k]')
subplot(2,1,2),stem(n,imag(X),'filled')
xlabel('k')
ylabel('FS Imaginary part')
title('imaginary part of X[k]')
figure(3)
subplot(2,1,1),stem(n,abs(X),'filled')
xlabel('k')
ylabel('amplitude')
title('Magnitude of X[k]')
subplot(2,1,2),stem(n,angle(X),'filled')
xlabel('k')
ylabel('amplitude')
title('Angle of X[k]')
figure(4)
subplot(2,1,1),stem(n,real(x1),'filled')
xlabel('time')
ylabel('Amplitude')
title('real part of x(n)')
subplot(2,1,2),stem(n,imag(x1),'filled')
xlabel('time')
ylabel('Amplitude')
title('imaginary part of x(n)')
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II- Numerical Approximation to Continuous Time Fourier Transform

A large class of signals can be represented using the continuous-time Fourier transform (CTFT). In this exercise we will use MATLAB to compute numerical approximations to the CTFT integral. By approximating the integral using a summation over closely spaced samples in $t$, you will be able to use the function $\text{ffti}$ to compute your approximation very efficiently. The approximation you will use follows from the definition of the integral:

$$
\int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt = \lim_{\tau \to \infty} \sum_{n=-\infty}^{\infty} x(n\tau) e^{-j\omega n\tau} \quad (2.1)
$$

For a large class of signals and for sufficiently small $\tau$, the sum on the right-hand side is a good approximation to the CTFT integral. If the signal $x(t)$ is equal to zero for and, then the approximation can be written as:

$$
\int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt \approx \int_0^{\tau} x(t) e^{-j\omega t} dt = \sum_{n=0}^{N-1} x(n\tau) e^{-j\omega n\tau} \quad (2.2)
$$

Where $T=N\tau$ and $N$ is an integer. You can use the function $\text{ffti}$ to compute the sum in Eqn. 2.2 for a discrete set of frequencies $\omega_k$. If the $N$ sampled $x(n\tau)$ are stored in the vector $x$, then the function call $x=\text{tau*ffti}$. calculates:

$$
X(K+1) = \tau \sum_{n=0}^{N-1} x(n\tau) e^{-j\omega_k \tau}, \quad 0 \leq k \leq N-1 \approx x(j\omega_k) \quad (2.3)
$$

Where

$$
\omega_k = \begin{cases} 
\frac{2\pi k}{N\tau}, & 0 \leq k \leq N/2 \\
\frac{2\pi (k-N)}{N\tau}, & \frac{N}{2} + 1 \leq k \leq N 
\end{cases}
$$

And $N$ is assumed to be even. For reasons of computational efficiency; $\text{ffti}$ returns the positive frequency samples before the negative samples. To place the frequency samples in ascending order, you can use the function $\text{ffshift}$. To store in $X$ samples of $x(j\omega_k)$ order such that $x(k+1)$ is the CTFT evaluated at $-\pi/\tau + (2\pi k/N\tau)$ for $0 \leq k \leq N-1$, use $x=\text{ffshift(fft(tau*ffti(x)))}$. For this exercise, you will approximate the CTFT of $x(t) = e^{-2|t|}$ using the $\text{ffti}$ and truncated version of $x(t)$. You will see that for sufficiently small $\tau$, you can compute an accurate numerical approximation of $x(j\omega)$.

a) Find an analytic expression for the CTFT of $x(t) = e^{-2|t|}$. You may find it helpful to think of $x(t) = g(t)+g(-t)$, where $g(t) = e^{-2|t|}u(t)$. 

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b) Create a vector containing samples of the signal \( y(t) = x(t-5) \) and \( \tau = 0.01 \) and \( T = 10 \) over the range \( t = [0:\tau : T - \tau] \). Since \( x(t) \) is effectively zero for \( |t| > 5 \), you can calculate the CTFT of the signal \( y(t) = x(t-5) \) from the above analysis using \( N = T / \tau \). Your vector \( y \) should have length \( N \).

c) Calculate samples \( Y(j\omega) \) by typing \( Y = \text{fftshift}(\tau \cdot \text{fft}(y)) \)

d) Construct a vector \( \omega \) of frequency samples that correspond to the values stored in the vector \( Y \) as follows:
\[
\omega = -\left(\frac{\pi}{\tau}\right) + (0: N - 1) \cdot (2 \cdot \pi / (N \cdot \tau))
\]
e) Since \( y(t) \) is related to \( x(t) \) through a time shift, the CTFT \( X(j\omega) \) is related to \( Y(j\omega) \) by a linear phase term of the form \( e^{j5\omega} \). Using the frequency vector \( \omega \), compute samples \( X(j\omega) \) directly from \( Y \), storing the result in the vector \( X \).

f) Using \textit{abs} and \textit{angle} plot the magnitude and phase of \( X \) over the frequency range specified in \( \omega \). For the same values of \( \omega \), also plot the magnitude and phase of the analytic expression you derived for \( X(j\omega) \) in part a. Does your approximation of the CTFT match what you derived analytically? If your plot the magnitude on a logarithmic scales using \textit{semilogy}, you will notice that at higher frequencies. Since you have approximated \( x(t) \) with samples \( x(n\tau) \), your approximation will be better for frequency components of the signal that do not vary much over time intervals of length \( \tau \).

g) Plot the magnitude and phase of \( Y \) using \textit{abs} and \textit{angle}. How do they compare with \( X \)? Could you have anticipated this result?

\textbf{Answer:}

tau=0.01;
T=10;
N=T/tau;
t1=[0:tau:T/2-tau];
t2=[T/2:tau:T-tau];
t=[t1 t2];
x=exp(-2*t2);
x1=exp(-2*t);
y=[zeros(1,500) x];
Y=fftshift(tau*fft(y));
X1=fftshift(tau*fft(x1));
w=-pi/tau+(0:N-1)*(2*pi/(N*tau));
x_analytically=4./(4+w.^2);
x1_analytically=1./(2+j*w);
y1_analytically=x1_analytically.*exp(-j*5*w);
figure(1)
plot(t,y)
xlabel('Time')
ylabel('Amplitude')
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title('y(t)')
figure(2)
plot(w,real(Y))
xlabel('Frequency')
ylabel('Amplitude')
title('Y(jw)')
figure(3)
plot(w,real(y1_analytically))
xlabel('Frequency')
ylabel('Amplitude')
title('Analytically expression of Y(jw)')
figure(4)
subplot(2,1,1),plot(w,abs(Y))
subplot(2,1,2),plot(w,angle(Y))
xlabel('Frequency')
ylabel('Amplitude')
title('Magnitude of y(jw)')
title('Phase of y(jw)')
figure(5)
subplot(2,1,1),plot(w,abs(x_analytically))
subplot(2,1,2),plot(w,angle(x_analytically))
xlabel('Frequency')
ylabel('Amplitude')
title('Analytically expression of Magnitude of X(jw)')
title('Analytically expression of Phase of X(jw)')
figure(6)
subplot(2,1,1),plot(w,abs(X1))
subplot(2,1,2),plot(w,angle(X1))
xlabel('Frequency')
ylabel('Amplitude')
title('Magnitude of X1(jw)')
title('Phase of X1(jw)')
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An analytic expression of $y(t)$

Magnitude of $Y(\omega)$

Phase of $Y(\omega)$

Magnitude of $X(\omega)$

Phase of $X(\omega)$